Scalar Product Equivalence

The dot/scalar product of two vectors $\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$ is *defined* to be

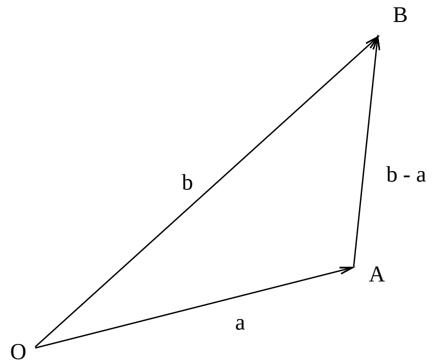
$$\mathbf{a}.\mathbf{b} \equiv |\mathbf{a}||\mathbf{b}|\cos\theta$$

where $|\mathbf{a}|$ denotes the magnitude (length) of the vector $\mathbf{a}$ and is calculated $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$ (similarly for $\mathbf{b}$) and θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$. This document is designed to demonstrate that the above definition is consistent with the component form of the dot product

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} . \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x b_x + a_y b_y + a_z b_z.$$

Proof

Consider the triangle OAB with $\theta = A\hat{O}B$.



Let $\overrightarrow{OA} = \mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ and $\overrightarrow{OB} = \mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$. Therefore $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} b_x - a_x \\ b_y - a_y \\ b_z - a_z \end{pmatrix}$.
Applying the cosine rule to the triangle we have

$$|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

so

$$\cos\theta = \frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{b} - \mathbf{a}|^2}{2|\mathbf{a}||\mathbf{b}|}$$

Therefore starting from the definition of the scalar product we find

$$\begin{aligned}
\mathbf{a}.\mathbf{b} &\equiv |\mathbf{a}||\mathbf{b}|\cos\theta \\
&= |\mathbf{a}||\mathbf{b}|\left(\frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{b} - \mathbf{a}|^2}{2|\mathbf{a}||\mathbf{b}|}\right) \\
&= \frac{1}{2}(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{b} - \mathbf{a}|^2) \\
&= \frac{1}{2}(a_x^2 + a_y^2 + a_z^2 + b_x^2 + b_y^2 + b_z^2 - ((b_x - a_x)^2 + (b_y - a_y)^2 + (b_z - a_z)^2)) \\
&= \frac{1}{2}(a_x^2 + a_y^2 + a_z^2 + b_x^2 + b_y^2 + b_z^2 - b_x^2 - a_x^2 + 2a_x b_x - b_y^2 - a_y^2 + 2a_y b_y - b_z^2 - a_z^2 + 2a_z b_z) \\
&= \frac{1}{2}(2a_x b_x + 2a_y b_y + 2a_z b_z) \\
&= a_x b_x + a_y b_y + a_z b_z.
\end{aligned}$$

As required.